

Quantum reconstruction of the mutual coherence function

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Light is a major carrier of information about the world around us, from the microcosmos to the macrocosmos. The present methods of detection are sensitive both to robust features, such as intensity, or polarization, and to more subtle effects, such as correlations. Here we show how wave front detection, which allows for registering the direction of the incoming wave flux at a given position, can be used to reconstruct the mutual coherence function when combined with some techniques previously developed for quantum information processing.

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Introduction.— Three-dimensional objects emit characteristic wave fronts that are determined by different object features, such as shape, refractive index, density, or temperature. Wave front sensing constitutes thus an invaluable tool to acquire information about our surroundings, especially because ordinary detection devices are not phase sensitive and only capture two-dimensional intensity data.

The measurement of the wave front phase distribution is a key issue for many applications, such as noncontact metrology, adaptive optics, high-power laser systems, and ophthalmology [1, 2]. Since the frequency of light waves is so high, no detectors of sufficient time resolution exist and indirect methods for phase measurements have been developed [3, 4], each one with their own pros and cons. In short, practical and robust wave front sensing is still an unresolved and demanding problem.

The purpose of this Letter is to point out that these standard methods may be underrated and do not fully exploit the potential of registered data. By recasting their functioning principle in a quantum language, one can immediately foresee that, by using the methods of tomographic reconstruction [5], these devices allow in fact for a full evaluation of the mutual coherence function of the signal, which conveys full information. This should be compared with the partial phase information retrieved by the standard wave front reconstruction techniques, where full coherence of the detected signal is assumed and imperfect correlations are ignored. Going beyond such a standard interpretation may constitute a substantial step ahead as it is indeed an interdisciplinary task involving wave and statistical optics, as well as protocols of quantum state reconstruction.

Classical theory of wave front measurements.— To be specific, we sketch the principle of a Hartmann-Shack sensor [6], which is general enough for our purposes here and is schematized in Fig. 1. To keep the discussion as simple as possible, we restrict ourselves to a one-dimensional model and denote by x the position in the scanning aperture. The incoming light field is divided into a number of subapertures by a microlens array that creates focal spots. The deviation of the spot pattern from a reference measurement allows the local direction angles to be derived, which in turn allows the reconstruction of the wave front. In addition, the light intensity distribution

at the detector plane can be obtained by integration and interpolation between the foci.

These devices provide a simultaneous detection of position x and angular spectrum p of the incident radiation, which is determined by the position of the detected signal on the screen. As we shall see below, p determines directly the transverse impulse p_x of incident radiation. Let Φ be the complex amplitude of the signal emitted by the source. The propagation of this signal is described by the convolution with the optical transfer function h [7], so that at the plane of the apertures it can be expressed as

$$\Phi_{\text{ap}}(x) = \int dx' \Phi(x') h(x' - x). \quad (1)$$

The amplitude in the focal plane due to the j th microlens of extension \mathcal{A}_j thus reads

$$\Phi_j(p) = \int_{\mathcal{A}_j} dx \Phi_{\text{ap}}(x) A_j(x) \exp\left(i \frac{k}{f} xp\right), \quad (2)$$

where $k = 2\pi/\lambda$ is the wave number, f the focal length, and the integration is performed in the transverse plane. The function A_j is the aperture (or pupil) function of the j th microlens.

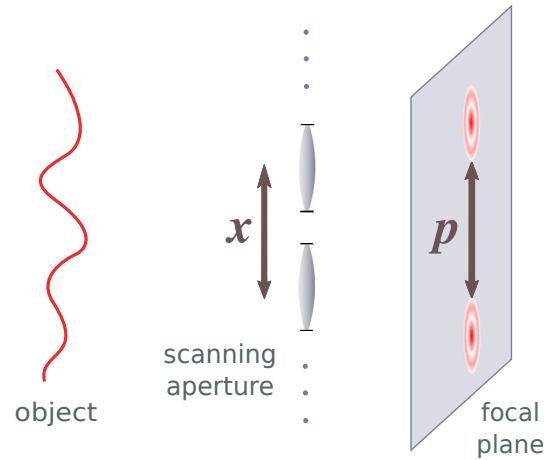


FIG. 1: Simplified scheme of a general scanning setup. Two of the microlenses of the periodic array are shown. By moving the aperture, the field is sampled at different positions, whereas by moving the detector in the focal plane the field is sampled at different angles.

Since only intensities can be detected, the corresponding signal at the focal plane is

$$S_{j,p} \equiv \langle |\Phi_j(p)|^2 \rangle = \int_{A_j} dx dx' dx'' dx''' Q(x, x') \times h(x - x'') h^*(x' - x''') \alpha_{j,p}(x'') \alpha_{j,p}^*(x'''). \quad (3)$$

Here $\langle \cdot \rangle$ denotes the averaging performed by the detection and $Q(x, x') = \langle \Phi(x) \Phi^*(x') \rangle$ is the mutual coherence function, which fully characterizes the source properties (up to second order). The combined factor $\alpha_{j,p}(x) = A_j(x) \exp(ikxp/f)$ condenses the lens focusing. Without loss of generality we can assume a smooth Gaussian approximation of this aperture profile, so that $A_j(x) = \exp[-\frac{1}{4}(x - x_j)^2/(\Delta x)^2]$, where x_j is the position of the center of the j th microlens and (Δx) is the Gaussian width, which can be considered identical for them all.

To show in a very simple manner the operating principle of these devices, let $\phi(x)$ represent the wave front under consideration. The mutual coherence function then reads $Q(x, x') = \exp\{ik[\phi(x') - \phi(x)]\}$. If $\phi(x)$ changes slowly at each microlens, it can be approximated by a linear expansion near the central point, so that $\phi(x) \simeq \phi(x_j) + (x - x_j)\phi'(x_j) \dots$, where $\phi'(x_j)$ is the derivative evaluated at x_j . Keeping the Gaussian approximation, the signal due to each microlens can be written, after a simple integration, as

$$S_{j,p} \simeq \exp\{-2k^2(\Delta x)^2 [p/f - \phi'(x_j)]^2\}. \quad (4)$$

The limit of geometrical optics can be characterized by the condition $k^2(\Delta x)^2 \rightarrow \infty$. By employing the identity $\lim_{\alpha \rightarrow \infty} \sqrt{\pi/\alpha} \exp(-\alpha t^2) = \delta(t)$, we immediately get

$$S_{j,p} \simeq \delta(p/f - \phi'(x_j)) \quad (5)$$

whose interpretation is quite convincing: the detection can be viewed as a local tilting of the wave front. This also shows that the displacement of the focal spot can be used to find the gradient of the surface $\phi(x)$, whence we can compute the wave front from a direct integration (or other adapted numerical schemes).

However, this is an approximation: the scheme depends also on the coherence of the signal. Loosely speaking, coherence expresses a mutual relation between two or more sources. Consider, for example, the imaging of two distant pointlike sources described, in the far field, by the plane waves $\exp(i\mathbf{k}_1 \cdot \mathbf{x})$ and $\exp(i\mathbf{k}_2 \cdot \mathbf{x})$ (see Fig. 2). Two limiting cases can be distinguished: for an incoherent superposition, the detector will record two spots indicating the two respective directions, while for coherent signals, the two plane waves will superpose resulting in a single wave with a modulated amplitude $2 \cos[(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{x}/2] \exp[i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}/2]$. In general, coherences in the signal will have a significant effect on the detected intensity profile. For example, when the two sources are separated by the Rayleigh distance the detected image consists of a single blurred point. In this particular case, two coherent point sources are not resolved by a single aperture, while incoherent sources at the same positions are just

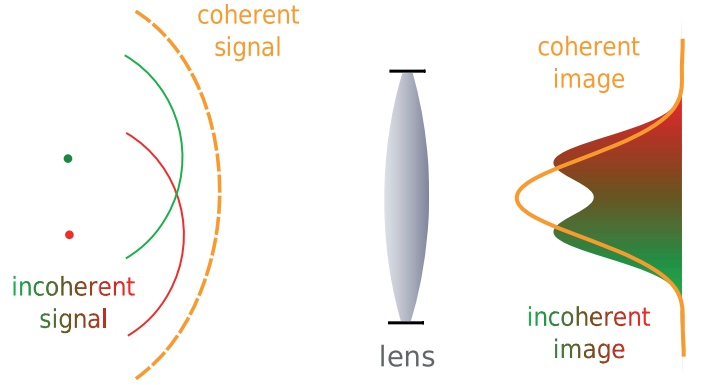


FIG. 2: Coherent (orange) and incoherent (red/green) images of two point sources separated by the Rayleigh distance. Notice how the recorded image varies with respect to the coherence properties of the signal.

resolved. In practice, the situation is even trickier: any realistic signal is always generated by a superposition of many plane waves, which may only be partially coherent.

Though the role of the coherence in image processing is clear, standard optical arguments always assume coherent or incoherent signals [7]. Partial coherence is mostly assumed in the context of interferometry and manifests by a reduction of the visibility. In fact, the celebrated van Cittert-Zernike theorem states the way in which coherence changes during the propagation.

Coherence is the main obstacle for the geometrical interpretation of the signal in a Shack-Hartmann sensor. Indeed, when registering the position and direction, there is still the uncertainty whether a single ray or several interfering rays were registered. This problem is solvable only at the level of the mutual coherence function. In view of these arguments the role of coherence seems to be underestimated in current data processing and it might provide new schemes. This is why we wish to bring into prominence the idea that the performance of optical scanning devices can be pushed up to the ultimate limits by resorting to tomographic reconstruction methods.

Quantum formulation.— We now make the important observation that the central equation (3) can be recast as

$$S_{j,p} = \langle \alpha_{j,p} | U^\dagger Q U | \alpha_{j,p} \rangle, \quad (6)$$

where we have introduced an obvious Dirac notation. The unitary transformation U accounts for the propagator. For example, employing the x representation, we have that $h(x - x') = \langle x | U | x' \rangle$, which gives a direct meaning to U . In addition, $Q(x, x') = \langle x | Q | x' \rangle$ plays the role of a density matrix and $\alpha_{j,p}(x) = \langle x | \alpha_{j,p} \rangle$. Equation (6) may appear as a simple reformulation of the problem; however, it is the basis for all our results: it can be seen as the optical interpretation of the quantum projection postulate. The scanning apparatus provides thus the same kind of information about the mutual coherence function as an ordinary measurement does about the state of a quantum system.

Note that special cases of (6) are the position intensity scan $I(x) = \langle x|Q|x \rangle$ and the spectral (angular) intensity scan $I(p) = \langle p|Q|p \rangle$, where $|p \rangle$ are the momentum (angular) states obtained from $|x \rangle$ by Fourier transformation. Therefore, the intensity $I(x)$ is a projection onto a precisely defined position, while the spectral intensity is a projection along the direction of the incoming wave. These intensities are measured by all optical devices with imaging capabilities, such as telescopes and photographic cameras.

To go beyond the standard image processing, we propose to reinterpret the operation of these devices as a generalized measurement of noncommuting variables. Indeed, position and momentum in the transverse plane cannot be measured simultaneously with arbitrary precision. In a quantum language, this represents a projection into a minimum uncertainty (squeezed) state, determined by the effective width of the Δx detection delimited by the microlens aperture. In terms of optics, the conjugate variable can be attributed to the impulse $p_x \simeq kp/f$, and fulfills the Heisenberg uncertainty principle $\Delta x \Delta p_x \geq \hbar/2$.

Simultaneous detection of noncommuting variables (such as position x and momentum p_x) is well known in quantum optics. This issue was addressed by Arthurs and Kelly [8] and can be properly formulated as a quantum estimation problem [9, 10]: one looks for the registered classical variables, let us call them X and P_x , that brings information about original quantum observables x and p_x . These new variables obey an uncertainty relation in which the lower bound is twice as large as the original Heisenberg uncertainty $\Delta X \Delta P_x \geq \hbar$.

The simultaneous measurement of x and p_x can be also modeled as the measurement of the non-Hermitian operator $A = x + ip_x$. Denoting the eigenstates of this operator as $|\alpha \rangle$, α being the complex number with the real part corresponding to the observable x and the imaginary part corresponding to p_x , the probability distribution for their simultaneous detection is the projection $P_H(\alpha) = \langle \alpha | \varrho | \alpha \rangle$, also known as the Husimi function [11] of the quantum state ϱ . Since P_H is fully equivalent to ϱ , the projections $|\alpha \rangle$ are informationally complete. In practice, however, P_H can be sampled only by a finite number of detections α and the tools of quantum state estimation [12–17] are especially germane to reconstruct ϱ from experimental data.

The strategy we suggest is straightforward:

- (i) The signal is measured with a wave front sensor consisting of a position sensitive detector placed in the focal plane of an array of microlenses. Alternatively a single lens can be moved with respect to the signal.
- (ii) To each pixel (of angular coordinate p) in the focal plane of the j th subaperture we associate a projection $|\alpha_{j,p} \rangle$ performed on the signal beam with the expectation value $S_{j,p}$. As we have shown above, in this way the signal reconstruction is converted to a quantum state reconstruction problem.
- (iii) The mutual coherence function Q of the signal is reconstructed by inverting the relation (6) with the constraint $Q \geq 0$. For example, the maximum likely mutual coherence matrix is obtained by iteratively solving the operator equation

$RQ = GQ$ [18], where

$$\begin{aligned} R &= \sum_{j,p} \frac{S_{j,p}}{S_{j,p}} |\alpha_{j,p} \rangle \langle \alpha_{j,p}|, \\ G &= \frac{\sum_{j,p} S_{j,p}}{\sum_{j,p} S_{j,p}} \sum_{j,p} |\alpha_{j,p} \rangle \langle \alpha_{j,p}| \end{aligned} \quad (7)$$

are positive semidefinite matrices, and $S_{j,p}$ are measured (noisy) data.

Once Q is known, any information of interest, such as, e.g., position or angular intensity scans in any transverse plane, coherence properties, etc., can be obtained from Q by postprocessing. Note also that the quantum aspects of the problem are manifested by the positive semidefiniteness constraint $Q \geq 0$.

Discussion.— The scanning devices of the type discussed above are not rare in Nature. For example, the compound eyes of insects [19] consist of thousands of identical units (ommatidia) and bear some similarity with a Hartmann-Shack sensor (see Fig. 3). As we have just argued, such a detector can, in principle, be used for reconstructing the mutual coherence function of the observed signal. This may have intriguing consequences for biology. For example, it is a commonly-accepted hypothesis that flies are short-sighted because the compound eye lacks the ability to accommodate and therefore to focus at different distances. Things may not be that simple: since the mutual coherence contains full information, focusing (or any other optical transformation) can be done by postprocessing the registered data and sharp images can be obtained without any need for optical focusing.

This can be clearly seen in Fig. 4, which presents an object consisting of a pair of spatially-separated bright spots. For a particular choice of the parameters, a single blurred spot is registered by a detector placed at the image plane of a single lens. A Hartmann-Shack device with just 10 subapertures fitted with 10-pixel detectors provides enough information to get a faithful reconstruction of the original object. We leave as an open question whether or not Nature actually takes advantage of the physics behind these scanning devices.

As the second example, where the coherence could be challenging are recent experimental results for temperature deviations of the cosmic microwave background. The anisotropy

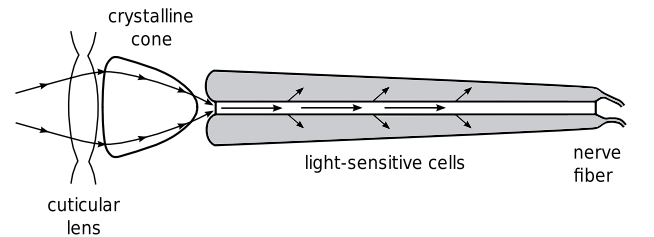


FIG. 3: Schematic drawing of a cut through an ommatidium of a fruit fly. The process of coupling the signal into the the light sensitive cell and the observed position, where the light is absorbed provide, in principle, information about the angular spectrum of the incident light. In this respect each ommatidium resembles a subaperture of a Hartmann-Shack sensor.

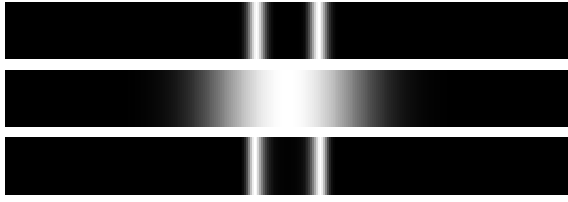


FIG. 4: Simulated reconstruction of a scanned optical signal. Panels from top to bottom: the simulated object, the blurred image registered by a short-sighted optical system, and the intensity distribution at the object plane as reconstructed from data provided by a scanning device.

is mapped as spots on the sphere [20], representing the distribution of directions of the incoming radiation. To get access to the position distribution, the detector has to be moved and, in principle, such a scanning sensor then brings information about the position and direction simultaneously. When the aperture is moving, it scans the field repeatedly at different positions, denoted here by the index j . Consequently, the registered signal expressed in a fixed reference frame reads $S_{j,p} = \langle \alpha_p | U_j^\dagger Q U_j | \alpha_p \rangle$.

The ideal blackbody radiation in thermal equilibrium should be homogeneous, isotropic and incoherent, as a consequence of Bose-Einstein statistics. However, recent missions of the projects COBE and WMAP have demonstrated convincingly that there is an anisotropy in the distribution of cosmic microwave background radiation. These deviations encode a wealth of information on the properties of the Universe in its infancy. The objective of the Planck mission is to measure these properties with unprecedented accuracy and level of details [21]. This could be also an excellent chance to investigate the subtle coherence properties of the relict radiation. To our best knowledge, this question has not been posed yet. Though the coherence cannot be measured directly by standard interferometric techniques, it could be inferred indirectly from the image patterns generated by partially coherent sources. The methods of quantum tomography are especially adequate for this task.

In summary, coherence can be used as an important source of information, provided that the mutual coherence is properly sampled by the detection. This possibility is not exploited in the standard protocols of image processing. The proposed statistical inversion, according to the receipts of quantum tomography, is the main result of this Letter. Notice that the acquired

mutual coherence function has the extraordinary property of recording all views of an object and the lack of resolution or accommodation could be more than compensated by the remarkable possibility of postprocessing this complete information, once it is accessible. This postprocessing is compatible with the laws of Nature, and can surely be a source of inspiration for the technology.

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